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Asymptotic Symmetry and the Global Structure of Future Null Infinity

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Space-times for which $\mathscr{I}^+$ (future null infinity) is not necessarily homeomorphic to $\mathbf{R}\times S^2$ are considered. It is shown that, depending on the global conformal structure of $\mathscr{I}^+$, a given space-time either (1) possesses an asymptotic symmetry group with a normal subgroup of supertranslations, similar in structure to the BMS group, or (2) possesses a simpler kind of asymptotic symmetry group, not involving supertranslations, or (3) has no asymptotic symmetry. The setting is Newman and Unti's approach to asymptotically flat space-times and use is made of the characterization of the asymptotic symmetry transformation as a conformal motion of $\mathscr{I}^+$ that preserves null angles.

1. INTRODUCTION

This paper is a sequel to an earlier one (Foster, 1978), which considered various approaches to asymptotic symmetry and interpreted them in terms of the conformal approach of Penrose. The space-times considered were those that satisfied the condition for asymptotic flatness proposed by Newman and Unti (1962), which is less restrictive than that usually adopted, in that it does not require $\mathscr{I}^+$ to be homeomorphic to $\mathbf{R}\times S^2$. By allowing more generality in the structure of $\mathscr{I}^+$, one can investigate how that structure determines the asymptotic symmetry. This seems to be the sort of question Newman and Unti raised in the closing remarks of the discussion in their paper, remarks that provided some of the motivation for the present paper. However, the main motivation was a desire to understand the origin of supertranslations, which are a feature of the BMS group (Bondi *et al.*, 1962; Sachs, 1962), the asymptotic symmetry group when $\mathscr{I}^+$ is homeomorphic to $\mathbf{R}\times S^2$.

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As is shown below, there are two extremes: one in which there is no asymptotic symmetry and the other in which $\mathscr{I}^+$ is conformal to a product manifold and the asymptotic symmetry group is a semidirect product, one factor of which comprises supertranslations. In between there is a variety of special cases where asymptotic symmetry groups exist, but none of these contains supertranslations.

The notation and understandings are those of the earlier paper (Foster, 1978), referred to henceforth as I. In particular, the coordinates for $\mathscr{I}^+$ are $x^0 = u$ and $x^i (i = 2, 3)$ [though the last two are soon replaced by the complex coordinate $z = \frac{1}{2}(x^2 + ix^3)$] and differentiation of $f(u, x^i)$ with respect to $x^A (A = 0, 2, 3)$ is denoted by $f_{,A}(u, x^i)$. In addition, it should be understood that an *asymptotic symmetry transformation* is a finite, nontrivial transformation, which may be derived from the identity in a continuous manner by integrating its infinitesimal generators. The *asymptotic symmetry group* formed by such transformations is therefore a continuous group.

Conformal transformations of three kinds enter into the discussion. The first is a *conformal transformation of the metric* whereby the metric tensor field is multiplied by a scalar field that, at each point of the manifold involved, is finite and nonzero (the singular behavior at $\mathscr{I}^+$ of the mapping $\tilde{\mathscr{M}} \to \mathscr{M}$ from the physical to the unphysical space-time being an exception). A property that does not change under such a transformation is *conformally invariant.* The second is a *conformal motion,* which is an angle-preserving diffeomorphism of a manifold onto itself, derivable from the identity in a continuous manner. The third is a *conformal mapping,* which is an angle-preserving bijection of a two-surface or a complex domain onto a two-surface or a complex domain. The statement that *A is conformal to B* will mean either that a conformal transformation of the metric of A yields that of B or that B is the image of A under a conformal mapping, the context making it clear which is meant. This terminology is used consistently throughout the paper.

2. ASYMPTOTIC TWO-SURFACES

As is shown in I (Section 2), the approach of Newman and Unti linked with the conformal approach of Penrose leads to the line element

$$ds_0^2 = \tfrac{1}{2}[P(u, x^k)]^{-2}\delta_{ij}\, dx^i\, dx^j \tag{1}$$

for $\mathscr{I}^+$. Setting $x^i = \text{const}$ gives a curve in $\mathscr{I}^+$ for which $ds_0^2 = 0$; that is, it is a null curve. Such curves, which are, in fact, null geodesics in the unphysical space $\mathscr{M}$, are the *generators* of $\mathscr{I}^+$, and u acts as a parameter along each generator. Setting $u = \text{const}$ gives an *asymptotic two-surface* $S(u)$, whose line element is also (1) and for which x^i act as isothermal coordinates.

A general asymptotic two-surface is given by setting $u = F(x^i)$, where F is arbitrary. The theory requires that $P(u, x^i)$ be a differentiable function of its arguments, so that each $S(u)$ has a differentiable metric tensor field. If this property is to be enjoyed by the general asymptotic two-surfaces, then F must be differentiable. The picture that emerges for $\mathscr{I}^+$ is that of a bundle of generators, a slice of which is an asymptotic two-surface.

The fact that the resulting two-surfaces possess isothermal coordinate systems means that certain regularity conditions have been assumed (see, for example, Bers, 1957). If one makes the further assumption that the two-surfaces are orientable, then they can be regarded as Riemann surfaces in a natural way. Thus, if $z = \frac{1}{2}(x^2 + ix^3)$ and $z' = \frac{1}{2}(x^{2'} + ix^{3'})$ are complex coordinates given by isothermal coordinates x^i and $x^{i'}$ (the inclusion of the factor one-half being for later convenience), then

$$z' = f(z) \tag{2}$$

where f is analytic, and, by restricting the coordinates to be isothermal, an analytic structure (in the complex sense) is given to each two-surface. [See Bers (1957) for details.] With the complex coordinate z replacing the x^i, (1) becomes

$$ds_0^2 = 2[P(u, z, \bar{z})]^{-2}\, dz\, d\bar{z} \tag{3}$$

The intention is to exploit the uniformization theory of Riemann surfaces to try to introduce a simple form for $P(u, z, \bar{z})$ and then to examine the asymptotic symmetry. However, there are problems: because of the dependence of P on u, what is simple for one value of u is not necessarily simple for any other.

The basic result of the uniformization theory of Riemann surfaces is that any simply connected Riemann surface may be mapped by a one-to-one analytic function onto exactly one of the following domains:

(a) The extended complex plane $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$.
(b) The complex plane $\mathbf{C}$.
(c) The open unit disk $D = \{z \mid |z| < 1\}$

In case (a) the surface is conformal to a sphere and is said to be *elliptic*, in case (b) it is conformal to a plane and is said to be *parabolic*, and in case (c) it is conformal to a hyperbolic plane (of which D is a model) and is said to be *hyperbolic*.

The domains $\mathbf{C}^*$, $\mathbf{C}$, and D may be thought of as *standard domains* and associated with them are the *standard line elements*

$$ds_0^2 = \frac{4\, dz\, d\bar{z}}{(1 + \kappa z\bar{z})^2} \tag{4}$$

of the unit sphere ($\kappa = 1$), the plane ($\kappa = 0$), and the unit hyperbolic plane ($\kappa = -1$). [The line element (4) is that of a surface of constant curvature κ.] However, for the subsequent discussion **C** and D are too special. If $\mathbf{C}^*$ is identified with the unit sphere in the usual way (i.e., by stereographic projection), then **C** may be regarded as a punctured sphere, the puncture caused by removing the point ∞. The more general domain in case (b) is a punctured sphere obtained by removing *any* point, and in case (c) is any simply connected domain having more than one boundary point. [For a verification of these and subsequent remarks about Riemann surfaces, see, for example, Springer (1957), in particular Chapter 9.]

It is worth noting here for later use that any conformal mapping of $\mathbf{C}^*$ onto itself is a Möbius transformation

$$z \to (az+b)/(cz+d) \tag{5}$$

(conventionally normalized by $ad - bc = 1$); that any conformal mapping of a punctured sphere onto a punctured sphere is given by a Möbius transformation mapping the puncture into the puncture, so that, in particular, a conformal mapping of **C** onto itself has the form

$$z \to az + b \tag{6}$$

and that any conformal mapping of D onto itself has the form

$$z \to (az+b)/(\bar{b}z+\bar{a}) \tag{7}$$

where $|a|^2 - |b|^2 = 1$. [Möbius transformations are extensively discussed in Schwerdtfeger (1962).]

Now, suppose for simplicity that each $S(u)$ is orientable and simply connected (but see Section 5 for a relaxation of the latter condition). Then by the above, each $S(u)$ is conformal to either (a) a sphere, (b) a plane, or (c) a hyperbolic plane, and which one it is will, in general, depend on u. The corresponding domain $D(u)$ of z will also depend on u. So one can picture a situation like that illustrated in Fig. 1, where, as u increases, $S(u)$ passes through a sequence of being conformal to a sphere, then a plane, then a hyperbolic plane, then a plane again, and finally a sphere again. [A dimension is, of course, suppressed. Slices are either circles, circles with a point removed, or circles with an arc removed, representing domains $D(u)$ corresponding to surfaces $S(u)$ that are conformal to a sphere, a plane or a hyperbolic plane, respectively. The straight line segments represent generators.]

A simple example is given by letting $S(u)$ be a (simply connected, orientable) surface with Gaussian curvature equal to $-u$, which can be achieved by setting

$$P(u, z, \bar{z}) = (1 - uz\bar{z})/\sqrt{2} \tag{8}$$

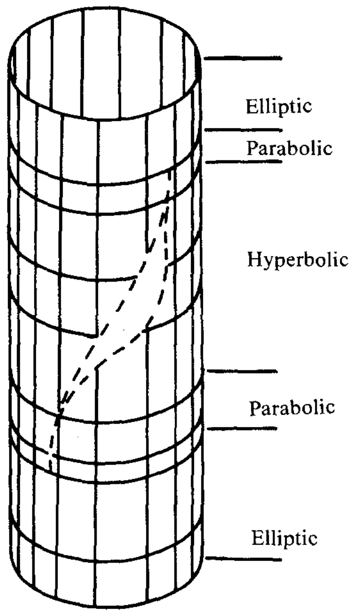

Fig. 1. Asymptotic two-surfaces.

Then, for $u<0$, $S(u)$ is a sphere and $D(u)$ is $\mathbf{C}^*$; for $u=0$, $S(u)$ is a plane and $D(u)=\mathbf{C}$; while for $u>0$, $S(u)$ is a surface of constant negative curvature and

$$D(u)=\{z||z|<1/\sqrt{u}\}.$$

3. NULL ANGLES AND CONFORMAL TRANSFORMATIONS OF THE METRIC

The main result of I is that an asymptotic symmetry transformation may be defined as a conformal motion of $\mathscr{I}^+$ that, in addition, preserves null angles, and this is the definition used here. The reason for favoring this definition is that it is conformally invariant, provided, of course, that the definition of equality of null angles is conformally invariant. In I (Section 3) the means of comparing null angles suggested by Penrose was replaced by a simpler working definition, which gave a way of measuring the size of a null angle in terms of the coordinates (u, x^i) and the quantity $P(u, x^i)$, and it is important to understand the sense in which this working definition is conformally invariant.

As explained in I (Section 2), to obtain $\mathscr{I}^+$ one goes through the following procedure. One first replaces Newman and Unti's radial coordinate r by $l=1/r$. Next one obtains the unphysical space-time $\mathscr{M}$ with line element ds^2 from the physical space-time $\tilde{\mathcal{M}}$ with line element $d\tilde{s}^2$ by means of the conformal transformation of the metric $ds^2 = l^2\, d\tilde{s}^2$. Finally, one sets $l=0$ in $\mathcal{M}$ to obtain $\mathcal{I}^+$ with line element ds_0^2 given by (1). Thus, in a precise way, the coordinate r determines the line element ds_0^2 of $\mathcal{I}^+$, and changing r induces a conformal transformation of the metric of $\mathcal{I}^+$. Equations (2.6) of I give the allowed coordinate transformations and show that a change in the coordinate u given by

$$u' = V_0(u, x^i) + O(r^{-1}) \tag{9}$$

induces the change in l given by

$$l' = V_{0,0}(u, x^i)l + O(r^{-2}) \tag{10}$$

which therefore induces the conformal transformation of the metric of $\mathcal{I}^+$ under which

$$ds_0^2 \to [V_{0,0}(u, x^i)]^2\, ds_0^2 \tag{11}$$

[Note that the second of equations (2.6) of I has an error. The correct equation is $r' = R_1(u, x^i)r + O(1)$.] The size of the null angle as defined in I is invariant under this induced conformal transformation of the metric provided that u is replaced by u' defined by

$$u' = V_0(u, x^i) \tag{12}$$

which is the form that (9) takes on $\mathcal{I}^+$, where $l=0$.

In short, to maintain conformal invariance in measuring null angles one should regard any conformal transformation of the metric of $\mathcal{I}^+$ as being generated by a change in the coordinate u, as given by (12), which induces the transformation (11) in the line element of $\mathcal{I}^+$. In the discussion below, such transformations are used to simplify $P(u, z, \bar{z})$.

4. ASYMPTOTIC SYMMETRY

As shown in I, a conformal motion of $\mathcal{I}^+$ is given by two functions V and Y that map the point with coordinates (u, z) to that with coordinates $(V(u, z, \bar{z}),\ Y(z))$. The function V is real-valued and differentiable, while Y is complex-valued and analytic. {See I, (2.16). That $Y(z) = \frac{1}{2}[Y^2(x^i) + iY^3(x^i)]$ is analytic follows from I, (2.17), which are the Cauchy-Riemann equations for Y.} To be an asymptotic symmetry transformation, the additional requirement that the motion preserve null angles must be satisfied. This means that V and Y are connected by the equation

$$P(V(u, z, \bar{z}),\ Y(z),\ \overline{Y(z)})\, V_{,0}(u, z, \bar{z}) = |Y'(z)| P(u, z, \bar{z}) \tag{13}$$

This was not given in I, but may be deduced by taking the definition of the size θ of a null angle as given in I (Section 3) (see I, Fig. 1), noting that under a conformal motion

$$|u_1 - u_2| \to V_{,0}(u, z, \bar{z})|u_1 - u_2|$$

and

$$d = \frac{2^{1/2}|dz|}{P(u, z, \bar{z})} \to \frac{2^{1/2}|Y'(z)||dz|}{P(V(u, z, \bar{z}), Y(z), \overline{Y(z)})}$$

(to first order of small quantities) and then imposing the condition that θ be preserved. [Alternatively, it may be deduced from I, (2.8) by putting $P' = P$ and changing from x^i to z.]

Under a conformal motion of $\mathscr{I}^+$ the generator z is mapped onto the generator $Y(z)$ and the asymptotic two-surface $S(u_0)$ is mapped onto that given by $u = V(u_0, z, \bar{z})$. Moreover, this mapping between two-surfaces is conformal (see I, Section 2). If $\mathscr{I}^+$ has a "tear" as in Fig. 1, then some generators are complete, while others are incomplete, and Y must not mix these. Also, slices of $\mathscr{I}^+$ yield two-surfaces of all three kinds (i.e., elliptic, parabolic, and hyperbolic) and V must be such that, for all u, $S(u)$ is mapped onto a surface of the same kind. These observations place severe restrictions on Y and V, which, together with the further condition that (13) be satisfied, suggest that asymptotic symmetry transformations do not exist, unless the structure of $\mathscr{I}^+$ is in some way special. It seems possible to compile a catalogue giving all the special configurations for which $\mathscr{I}^+$ possesses asymptotic symmetry, but there seems little to be gained from completing the task. The following discussion is confined to a few configurations sufficient to illustrate the sort of asymptotic symmetry groups that can arise and to facilitate the discussion of supertranslations. In this section all asymptotic two-surfaces are simply connected, but in the next, brief consideration is given to relaxing this condition.

4.1. All Two-Surfaces Elliptic

Each $S(u)$ is conformal to a sphere, so $D(u) = \mathbf{C}^*$ for all u, and [from (3) and (4)]

$$P(u, z, \bar{z}) = J(u, z, \bar{z})(1 + z\bar{z})/\sqrt{2} \tag{14}$$

where $J(u, z, \bar{z})$ is finite and nonzero for all points in $\mathscr{I}^+$. A conformal transformation of the metric of $\mathscr{I}^+$ (generated by a change in the coordinate u, as explained in Section 3) can be used to make $J(u, z, \bar{z}) = 1$, so that

$$P(u, z, \bar{z}) = (1 + z\bar{z})/\sqrt{2} \tag{15}$$

The only restriction on Y is that it should map $\mathbf{C}^*$ onto itself. Hence it is a Möbius transformation,

$$Y(z) = (az+b)/(cz+d) \tag{16}$$

where, without loss of generality, $ad - bc = 1$. In order that (13) be satisfied,

$$V(u, z, \bar{z}) = K(z, \bar{z})u + \alpha(z, \bar{z}) \tag{17}$$

where

$$K(z, \bar{z}) = Z^*Z/Z^*\Lambda^*\Lambda Z \tag{18}$$

and here

$$\Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad Z = \begin{bmatrix} z \\ 1 \end{bmatrix} \tag{19}$$

(and a star denotes the conjugate transpose). The function α is real-valued, differentiable, but otherwise arbitrary, with domain $\mathbf{C}^*$.

The transformation described here is, of course, a BMS transformation with $\alpha(z, \bar{z})$ giving the supertranslation part.

4.2. All Two-Surfaces Parabolic

Each $S(u)$ is conformal to a plane, so each $D(u)$ is a punctured sphere:

$$D(u) = \mathbf{C}^* \backslash \{\omega(u)\} \tag{20}$$

where $\omega(u)$ is a point of $\mathbf{C}^*$. It is convenient to treat the case where $\omega(u)$ is constant separately from those in which it varies.

4.2.1. $\omega(u)$ Constant

There is no loss of generality in having $\omega(u) = \infty$, so each $S(u)$ has the standard domain $D(u) = \mathbf{C}$, and [from [3] and (4)]

$$P(u, z, \bar{z}) = J(u, z, \bar{z})/\sqrt{2} \tag{21}$$

where $J(u, z, \bar{z})$ is finite and nonzero for all points in $\mathscr{I}^+$. Again, by a conformal transformation of the metric, one can make $J(u, z, \bar{z}) = 1$, so that

$$P(u, z, \bar{z}) = 1/\sqrt{2}$$

The restriction on Y is that it should map $\mathbf{C}$ onto itself, and it is therefore an affine transformation,

$$Y(z) = az + b \tag{22}$$

In order that (13) be satisfied,

$$V(u, z, \bar{z}) = |a|u + \alpha(z, \bar{z}) \tag{23}$$

where α is a real-valued, differentiable, but otherwise arbitrary function with domain **C**.

The transformation is similar to a BMS transformation, with $\alpha(z, \bar{z})$ giving a supertranslation.

4.2.2. $\omega(u)$ Not Constant

A suitable form for $P(u, z, \bar{z})$ can be obtained by noting that, for each value of u, there is a Möbius transformation (depending on u) that maps $\omega(u) \to \infty$ and yields a line element that is $J(u, z, \bar{z})$ times the standard line element (4) (with $\kappa = 0$), where the factor $J(u, z, \bar{z})$ can be set equal to unity by a conformal transformation of the metric of $\mathscr{I}^+$. If this transformation is

$$z \to \frac{a(u)z + b(u)}{c(u)[z - \omega(u)]} \tag{24}$$

where

$$-a(u)c(u)\omega(u) - b(u)c(u) = 1$$

then the condition that it yield the standard metric implies that

$$P(u, z, \bar{z}) = |c(u)|^2 |z - \omega(u)|^2 / \sqrt{2} \tag{25}$$

There is still some freedom to adjust $P(u, z, \bar{z})$ by choosing $c(u)$ appropriately, and this will be exploited to give simple expressions for the asymptotic symmetry transformations.

The set of punctures forms a curve in $\mathscr{I}^+$ (for P being differentiable requires that ω be differentiable and therefore continuous) and its projection (via generators) gives a curve γ in $\mathbf{C}^*$, given parametrically by $u \to \omega(u)$. If $(u, z) \to (V(u, z, \bar{z}), Y(z))$ is an asymptotic symmetry transformation, then Y must be a Möbius transformation leaving γ invariant, since, for each u, the puncture $\omega(u)$ that determines the domain $D(u)$ of $S(u)$ must be mapped into the puncture determining the domain of the image of $S(u)$. For a general curve, there is no such Möbius transformation other than the identity. If $Y(z) = z$, then applying condition (13) to P as given by (25) results in $V(u, z, \bar{z}) = u$, showing that, for a general γ, no asymptotic symmetry transformation exists. However, certain curves are invariant under Möbius transformations and these lead to asymptotic symmetry groups. There are essentially two cases to consider, according as the Möbius transformation has one or two fixed points.

Suppose first that $z \to Y(z)$ has one fixed point. Without loss of generality this can be taken to be ∞, and then Y is simply a translation

$$z \to z + C \qquad (C \text{ complex})$$

which leaves invariant any straight line with direction C. This leads to the following special case.

Case A. If

$$\omega(u) = Au + \omega_0 \qquad (A, \omega_0 \text{ complex constants})$$

and if, in (25), $|c(u)| = 1$ [which can be achieved by taking $a(u) = 0$, $-b(u) = c(u) = 1$ in (24)], then, for all $\alpha \in \mathbf{R}$,

$$V(u, z, \bar{z}) = u + \alpha, \qquad Y(z) = z + A\alpha \tag{26}$$

is an asymptotic symmetry transformation [as can be checked using (12) and (25)].

On the other hand, if $z \to Y(z)$ has two fixed points, then taking these to be 0 and ∞ results in

$$z \to Cz \qquad (C \text{ complex})$$

which represents a dilatation with scale factor $|C|$ combined with a rotation through an angle arg C. This leaves invariant curves that are logarithmic spirals (including circles if $|C| = 1$ and straight lines if arg $C = 0$) and results in the following special case.

Case B. If

$$\omega(u) = \omega_0 A^u \qquad (A, \omega_0 \text{ complex constants})$$

and if, in (25), $|c(u)|^2 = 1/|\omega(u)|$ {which can be achieved by taking $-a(u) = c(u) = [\omega(u)]^{-1/2}$, $b(u) = 0$ in (24)}, then, for all $\alpha \in \mathbf{R}$,

$$V(u, z, \bar{z}) = u + \alpha, \qquad Y(z) = A^\alpha z \tag{27}$$

is an asymptotic symmetry transformation (as is readily checked).

In either case the asymptotic symmetry transformations [as given by (26) or (27)] form a group isomorphic to the additive group of real numbers $(\mathbf{R}, +)$.

4.3. All Two-Surfaces Hyperbolic

Each $S(u)$ is conformal to a hyperbolic plane, so each $D(u)$ is a simply connected domain whose boundary has more than one point. As with a parabolic two-surface, it is convenient to consider the case where $D(u)$ is constant separately from those in which is varies with u.

4.3.1. D(u) Constant.

The constant domain $D(u)$ can be mapped onto the standard domain $D = \{z \,|\, |z| < 1\}$, so that D becomes the domain for all $S(u)$. Then [from (3) and (4)]

$$P(u, z, \bar{z}) = J(u, z, \bar{z})(1 - z\bar{z})/\sqrt{2} \tag{28}$$

where $J(u, z, \bar{z})$ is finite and nonzero for all points in $\mathscr{I}^+$, and by a conformal transformation of the metric of $\mathscr{I}^+$ one can make $J(u, z, \bar{z}) = 1$, giving

$$P(u, z, \bar{z}) = (1 - z\bar{z})/\sqrt{2} \tag{29}$$

Since Y must map D onto itself, it has the form

$$Y(z) = (az + b)/(\bar{b}z + \bar{a}) \tag{30}$$

where $|a|^2 - |b|^2 = 1$, and, in order that (12) be satisfied,

$$V(u, z, \bar{z}) = u + \alpha(z, \bar{z}) \tag{31}$$

where α is a real-valued, differentiable, but otherwise arbitrary function with domain D.

The transformation is similar to a BMS transformation, with $\alpha(z, \bar{z})$ giving a supertranslation.

4.3.2. D(u) Not Constant.

The situation here is more complicated than the corresponding situation in the case of parabolic two-surfaces. The reason is that it is not possible to give a general form for $P(u, z, \bar{z})$ analogous to (25), due to the fact that for a given general $D(u)$ there is no simple analogue of (24). [The analogue is a transformation involving a parameter u, which, for each value of u, maps $D(u)$ onto the unit disk D.] However, it is clear that there will be no asymptotic symmetry unless $D(u)$ is special.

Suppose, for example, that for some choice of the coordinate z, each $D(u)$ is a circle, with center $\sigma(u)$ and radius $\rho(u)$, say. Then

$$z \to [z - \sigma(u)]/\rho(u) \tag{32}$$

maps $D(u)$ onto D and yields the standard line element (4) (with $\kappa = -1$) if

$$P(u, z, \bar{z}) = \{[\rho(u)]^2 - |z - \sigma(u)|^2\}/2^{1/2}\rho(u) \tag{33}$$

All transformations from $D(u)$ onto D yield (33) and of these (32) is the simplest to write down. There is no freedom to adjust $P(u, z, \bar{z})$ by adjusting the transformation, like that embodied in (25), where $|c(u)|^2$ can be varied. There is, of course, freedom to multiply $P(u, z, \bar{z})$ by a factor $J(u, z, \bar{z})$, corresponding to a conformal translation of the metric of $\mathscr{I}^+$, but this will not be used, as (33) yields convenient expressions in the special cases below.

Case A. If

$$\sigma(u) = Au + \sigma_0, \qquad \rho(u) = \rho_0$$

(A, σ_0 complex constants; ρ_0 real, positive constant), then, for all $\alpha \in \mathbf{R}$,

$$V(u, z, \bar{z}) = u + \alpha, \qquad Y(z) = z + A\alpha \tag{34}$$

is an asymptotic symmetry transformation.

Case B. If

$$\sigma(u)=\sigma_0 A^u, \qquad \rho(u)=\rho_0|A|^u$$

(A, σ_0 complex constants; ρ_0 real, positive constant), then, for all $\alpha \in \mathbf{R}$,

$$V(u, z, \bar{z})=u+\alpha, \qquad Y(z)=A^\alpha z \tag{35}$$

is an asymptotic symmetry transformation.

Case C. If $\sigma(u)=0$ and $\rho(u)$ is arbitrary, then, for all $\theta \in \mathbf{R}$,

$$V(u, z, \bar{z})=u, \qquad Y(z)=e^{i\theta}z \tag{36}$$

is an asymptotic symmetry transformation.

In cases A and B, the asymptotic symmetry groups are isomorphic to $(\mathbf{R}, +)$, and each is similar to the corresponding special case where the two-surfaces are parabolic. However, case C has no parabolic counterpart and the group here is isomorphic to $SO(2, \mathbf{R})$. These three cases do not exhaust all possibilities: a fourth special case has $\sigma(u)=0$ and $\rho(u)=\rho_0 C^u$ (C real and positive) and yields a group isomorphic to $(\mathbf{R}, +)\times SO(2, \mathbf{R})$.

4.4. Two-Surfaces of All Three Kinds

It is not necessary that the asymptotic two-surfaces be all of the same kind for asymptotic symmetry transformations to exist. The example at the end of Section 2 with $P(u, z, \bar{z})$ given by (8) is like this and, for all $\theta \in \mathbf{R}$,

$$V(u, z, \bar{z})=u, \qquad Y(z)=e^{i\theta}z \tag{37}$$

is an asymptotic symmetry transformation. Such transformations form a group isomorphic to $SO(2, \mathbf{R})$.

5. TOPOLOGICAL IDENTIFICATIONS

A Riemann surface S is not in general simply connected, but its universal covering surface $\hat{S}$ is. One constructs S from $\hat{S}$ by identifying points that are equivalent under elements of a covering group Γ [see Springer (1957), in particular Chapter 9]. This idea may be extended to $\mathscr{I}^+$ by taking the simply connected two-surfaces $S(u)$ of Sections 2 and 4 to be covering surfaces $\hat{S}(u)$ and introducing a family of covering groups $\Gamma(u)$, each of which has elements that identify points in $\hat{S}(u)$ to yield a two surface $S(u)$ that is not simply connected. The claim is that, for the special cases presented in Section 4, it is possible to introduce covering groups $\Gamma(u)$ in such a way that the asymptotic symmetry is retained. While the discussion here does not consider all cases, sufficient is done to illustrate the idea and to lend support to the claim.

The cases most readily disposed of are those in which the domain $D(u)$ of $\hat{S}(u)$ does not depend on u. There are three such cases: that of elliptic two-surfaces discussed in Section 4.1, that of parabolic two-surfaces discussed in Section 4.2.1, and that of hyperbolic two-surfaces discussed in Section 4.3.1. As argued above, one can arrange things so that

$$P(u, z, \bar{z}) = (1 + \kappa z\bar{z})/\sqrt{2} \tag{38}$$

where $\kappa = 1$, 0, or -1, and it can be seen that the space-times involved are just the ones having the DS-spaces of Robinson and Trautman (1962) as prototypes. The question of introducing topological identifications into the DS-spaces and its effect on asymptotic symmetry was discussed in an earlier paper (Foster, 1969). The results for these prototypes [summarized in Table 2 of Foster (1969)] hold for the more general space-times considered here.

For example, in the parabolic case, where each $\hat{S}(u)$ is conformal to a plane and each $D(u) = \mathbf{C}$, the covering group $\Gamma(u)$ consisting of transformations

$$z \to z + m\lambda \tag{39}$$

where $\lambda \in \mathbf{C}$ and $m = 0, \pm 1, \pm 2, \ldots$, identifies points in $\hat{S}(u)$ whose coordinates differ by a multiple of λ. This has the effect of rolling up $\hat{S}(u)$ to obtain $S(u)$ as a cylinder. For equations (22) and (23) to represent an asymptotic symmetry transformation of $\mathscr{I}^+$ with cylindrical two-surfaces $S(u)$, a further condition must be satisfied: identified points must be mapped into identified points. This condition is satisfied if Y [as given by (22)] commutes with each element of $\Gamma(u)$ [as given by (39)] and if $\alpha(z, \bar{z})$ [occurring in (23)] satisfies

$$\alpha(z + m\lambda, \bar{z} + m\bar{\lambda}) = \alpha(z, \bar{z}) \tag{40}$$

for all $m = 0, \pm 1, \pm 2, \ldots$. The commutativity requirement means that in (22) $a = 1$, so an asymptotic symmetry transformation has the form

$$\begin{aligned} V(u, z, \bar{z}) &= u + \alpha(z, \bar{z}) \\ Y(z) &= z + b \end{aligned} \tag{41}$$

where α satisfies (40).

In this example, the covering group $\Gamma(u)$ does not depend on u, and the construction of $S(u)$ from $\hat{S}(u)$ is the same for all u. This is typical of all cases where $D(u)$ is constant and where $P(u, z, \bar{z})$ can be brought to one of the forms (38). However, if $D(u)$ is not constant, then any $\Gamma(u)$ will depend on u and the situation is more complicated.

Consider, for example, case B of Section 4.2.2, where each two-surface $\hat{S}(u)$ is a punctured sphere and the curve of punctures is a spiral [given by $\omega(u) = \omega_0 A^u$], and suppose one wishes to introduce $\Gamma(u)$ so that each $\hat{S}(u)$

is rolled up into a cylinder $S(u)$, but in such a way that asymptotic symmetry is retained. In place of (39) one has the Möbius transformation $\gamma(m, u)$ defined by

$$z \to \frac{(1-m\lambda)z+(\omega_0 A^u m\lambda)}{(-\omega_0^{-1}A^{-u}m\lambda)z+(1+m\lambda)} \tag{42}$$

This form for $\gamma(m, u)$ can be obtained by mapping $D(u) \to \mathbf{C}$ by means of (24) with $-a(u)=c(u)=[\omega(u)]^{-1/2}$ and $b(u)=0$. Note that $\gamma(m, u)$ maps $\omega(u)$ into itself, as it should. The asymptotic symmetry transformation (27) remains an asymptotic symmetry transformation without further restriction, because it maps identified points into identified points. The last assertion follows from the fact that, for all m and u, the identity

$$Y \circ \gamma(m, u)=\gamma(m, u+\alpha) \circ Y \tag{43}$$

holds, as is readily checked using (27) and (42). (This replaces the commutativity requirement of the previous example.)

In a similar way one can give covering transformations that roll each $\hat{S}(u)$ up into a cylinder in case A of Section 4.2.2, where the curve of punctures is a straight line. They have the form

$$z \to \frac{[1-(Au+\omega_0)m\lambda]z+(Au+\omega_0)^2 m\lambda}{(-m\lambda)z+[1+(Au+\omega_0)m\lambda]} \tag{44}$$

Equations (26) continue to represent an asymptotic covering transformation without further restriction.

The other kind of multiply connected surface that has a parabolic covering surface is a torus, and the covering transformations (39), (42), and (44) are easily adapted so that they identify points in $S(u)$ to yield a torus for $S(u)$: one simply replaces $m\lambda$ by $m\lambda+n\mu$ ($\lambda, \mu \in \mathbf{C}$; $m, n=0, \pm 1, \pm 2, \ldots$).

The handling of asymptotic two-surfaces having hyperbolic covering surfaces is somewhat more complicated and is not considered here.

6. SUMMARY AND DISCUSSION

If the generators of $\mathscr{I}^+$ are complete, then $D(u)$ does not depend on u. By taking $D(u)$ to be a standard domain (or a subset forming a fundamental domain, if identifications under a covering group are allowed) one has

$$P(u, z, \bar{z})=J(u, z, \bar{z})(1+\kappa z\bar{z})/\sqrt{2} \tag{45}$$

where $\kappa=0, \pm 1$ and $J(u, z, \bar{z})$ is finite and nonzero for all points of $\mathscr{I}^+$, and by means of a conformal transformation of the metric (generated by changing the coordinate u, as explained in Section 3) one can make $J(u, z, \bar{z}) = 1$ so that $P(u, z, \bar{z})$ takes on a standard form

$$P(z, \bar{z}) = (1 + \kappa z\bar{z})/\sqrt{2} \tag{46}$$

which does not depend on u. If $(u, z) \to (V(u, z, \bar{z}), Y(z))$ is an asymptotic symmetry transformation, then Y is restricted to mapping the standard domain onto itself (and to commuting with any covering transformations) and $V_{,0}(u, z, \bar{z})$ is then determined in terms of $P(z, \bar{z})$ and $Y(z)$:

$$V_{,0}(u, z, \bar{z}) = F(z, \bar{z}) \tag{47}$$

where

$$F(z, \bar{z}) = |Y'(z)| P(z, \bar{z}) / P(Y(z), \overline{Y(z)}) \tag{48}$$

Hence

$$V(u, z, \bar{z}) = F(z, \bar{z})u + \alpha(z, \bar{z}) \tag{49}$$

where α is arbitrary, and in this way the transformation is seen to be determined by Y and α.

If $Y(z) = z$, then the asymptotic symmetry transformation reduces to

$$(u, z) \to (u + \alpha(z, \bar{z}), z) \tag{50}$$

which is a *supertranslation.* A general asymptotic symmetry transformation can be regarded as a transformation of the form

$$(u, z) \to (F(z, \bar{z})u, Y(z)) \tag{51}$$

where $F(z, \bar{z})$ is given by (48), followed by a supertranslation (50). It is a straightforward matter to verify that the asymptotic symmetry transformations form a group ASG (the *asymptotic symmetry group* of the particular $\mathscr{I}^+$ under consideration), that the supertranslations form a normal subgroup ST of ASG, and that the transformations of the form (51) form a group CT isomorphic to one of classical type [e.g., $SL(2, \mathbf{C})$]. In fact, $ST \rtimes CT = ASG$, where $\rtimes$ denotes the semidirect product.

The essential points to note are (1) if the generators of $\mathscr{I}^+$ are complete, then $D(u)$ is constant and one can arrange things so that $P(u, z, \bar{z})$ does not depend on u, (2) this means that $\mathscr{I}^+$ is conformal to a product manifold, and (3) an asymptotic symmetry group exists and has a normal subgroup of supertranslations. The use of standard domains and standard forms for $P(z, \bar{z})$ is not essential, but leads to simple forms for the asymptotic symmetry transformations.

On the other hand, if $\mathscr{I}^+$ has incomplete generators, then it is impossible to arrange things so that $P(u, z, \bar{z})$ is independent of u and in general no asymptotic symmetry group exists. However, there may be special configurations (like those discussed in Section 4) where asymptotic symmetry groups do exist, though none of these contains supertranslations. In their original paper, Newman and Unti give a transformation that rids $P(u, z, \bar{z})$ of its u dependence. Clearly, unless $\mathscr{I}^+$ is conformal to a product manifold, such a transformation can have only local validity. As remarked in I, ridding $P(u, z, \bar{z})$ of its u dependence is equivalent to ridding the generators of their divergence.

It can be seen that a supertranslation represents the freedom to map each generator of $\mathscr{I}^+$ onto itself while preserving null angles. For a coordinate u that yields a form for $P(u, z, \bar{z})$ independent of u, this map has the form $u \to u + \alpha(z, \bar{z})$, giving a translation of each generator onto itself, but, because of the arbitrariness in $\alpha(z, \bar{z})$, this translation varies from generator to generator. It is clear from the way in which null angles are measured (see I, Section 3) why, with $P(u, z, \bar{z})$ independent of u, this arbitrary sliding of the generators into themselves preserves null angles.

Stipulating that $\mathscr{I}^+$ is homeomorphic to $\mathbf{R} \times \mathbf{S}^2$ implies that each asymptotic two-surface is elliptic, which is sufficient to deduce that $\mathscr{I}^+$ is conformal to a product manifold and therefore that there exists an asymptotic symmetry group, complete with supertranslations, namely the *BMS* group, as explained in Section 4.1. However, it is only in this case that the topological structure of $\mathscr{I}^+$ alone is sufficient to determine its conformal structure and therefore its asymptotic symmetry group. For example, stipulating that $\mathscr{I}^+$ is homeomorphic to $\mathbf{R}^3$, implying that each asymptotic two-surface is homeomorphic to $\mathbf{R}^2$, is insufficient, for that admits both parabolic and hyperbolic two-surfaces. Even if one were more precise and stipulated in addition that, for example, each two-surface were parabolic, then still the asymptotic symmetry is not determined, and one must explain how the two-surfaces are put together so as to yield the overall conformal structure of $\mathscr{I}^+$ and thereby distinguish between the various cases considered in Section 4.2.

The approach to the conformal structure of $\mathscr{I}^+$ adopted here is based on treating the asymptotic two-surfaces as Riemann surfaces and leads to a model of $\mathscr{I}^+$ (in the case where the two-surfaces are simply connected) as a subset of $\mathbf{R} \times \mathbf{C}^*$. If the two-surfaces are not all elliptic, then this is a proper subset having a boundary (as illustrated in Fig. 1). Any conformal motion of $\mathscr{I}^+$ onto itself must leave this boundary invariant, and unless this boundary possesses some sort of symmetry, then no such motion will exist. This is essentially why there is in general no asymptotic symmetry in the cases where $\mathscr{I}^+$ is not homeomorphic to $\mathbf{R} \times \mathbf{S}^2$. The exceptional cases are of two kinds. The first has the boundary of $\mathscr{I}^+$ comprising generators, so that it is conformal to a product manifold and has an asymptotic symmetry group with a subgroup of supertranslations. The second has the boundary of $\mathscr{I}^+$ invariant under a special motion such as the sliding motions represented by (26) and (34), the screwing motions represented by (27) and (35), and the rotational motions represented by (36) and (37). The asymptotic symmetry group then has a simple structure without supertranslations.

It has been argued that any reasonable definition of asymptotic flatness should have $\mathscr{I}^+$ homeomorphic to $\mathbf{R}\times\mathbf{S}^2$ (Penrose, 1965; Persides, 1979) and that the space-times admitted by Newman and Unti's original definition are not necessarily asymptotically flat, despite their having the correct "falloff" behavior as one goes out to infinity in a given null direction (Ludwig, 1981). If the space-times are to represent radiation from isolated sources, then the argument in favor of $\mathbf{R}\times\mathbf{S}^2$ is strong and no alternative is being suggested here. Indeed, one can give support to that argument as follows.

The *DS*-spaces of Robinson and Trautman have line elements

$$ds^2=(\kappa-2m/r)\,du^2+2\,du\,dr-4r^2(1+\kappa z\bar{z})^{-2}\,dz\,d\bar{z} \tag{52}$$

where m is a nonzero constant and $\kappa=\pm1$ or 0, and for these $\mathscr{I}^+$ is a product manifold with asymptotic two-surfaces that are elliptic if $\kappa=1$, parabolic if $\kappa=0$, and hyperbolic if $\kappa=-1$ (Foster, 1969). Each of these space-times has a well-defined, flat-space-time, electromagnetic analogue. That corresponding to $\kappa=1$ is singular along a timelike line, being the Coulomb field of a point charge, which is the electromagnetic analogue of the Schwarzschild field. However, the electromagnetic fields corresponding to $\kappa=0$ and $\kappa=-1$ are singular on null hypersurfaces, both of which extend to spatial infinity (Foster, 1971). Hence, only in the case where $\kappa=1$ is the source of the electromagnetic field spatially bounded. Thus, via the Robinson–Trautman prototypes and their electromagnetic analogues, one can argue that in order to represent radiation from isolated sources, the asymptotic two-surfaces of $\mathscr{I}^+$ should be elliptic, i.e., that $\mathscr{I}^+$ should be homeomorphic to $\mathbf{R}\times\mathbf{S}^2$.

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